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ABSTRACT

Design and results of an investigation attempting to analyze and improve children's solution processes in elementary addition and subtraction problems are described. As background for the study, a conceptual model was developed based on previous research. One dimension of the model relates to the characteristics of the tasks (numerical versus word problems, direct versus indirect, etc.). The second dimension has to do with the problem-solving process that is conceived in terms of three stages: thinking phase, technical or executive phase, and control phase. Using error analysis and individual interviews, the action structures (appropriate as well as inappropriate) of first and second graders were analyzed. On the basis of this ascertaining study, important shortcomings of children's knowledge and solution strategies were discovered. It also seemed that these shortcomings could be overcome by instruction; therefore, a teaching experiment was undertaken wherein systematic instruction was given for two weeks to an experimental class, while in a control group the usual arithmetic program was taught. Experimental instruction related mainly to the following three topics: an exact notion of the equality sign (=), the whole-part relation, and the action of controlling the outcome of an arithmetic operation. Results of the experiment are discussed. (Author/MK)

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Report no. 19

CHILDREN'S SOLUTION PROCESSES IN ELEMENTARY ARITHMETIC PROBLEMS : ANALYSIS AND IMPROVEMENT

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CHILDREN'S SOLUTION PROCESSES IN ELEMENTARY ARITHMETIC PROBLEMS : ANALYSIS AND IMPROVEMENT

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Summary

In this paper, the design and results of an investigation are described wherein we have tried to analyze and improve children's solution processes in elementary addition and subtraction problems such as the missing addend problem.
As background for the study, a conceptual model was developed based on previous research. One dimension of the model relates to the characteristics of the tasks (numerical versus word problems, direct versus indirect problems, etc.). The second dimension has to do with the problem-solving process that is conceived in terms of three stages : a thinking phase, a technical or executive phase, and a control phase. Using error analysis and individual interviews, the action structures (appropriate as well as inappropriate) of first and second graders were analyzed.
On the basis of this ascertaining study, important shortcomings of children's knowledge and solution strategies were discovered. It also seemed that these shortcomings could be overcome by instruction; therefore, a teaching experiment was undertaken wherein systematic instruction was given for two weeks to an experimental class while, in a control group, the usual arithmetic program was taught. Experimental instruction related mainly to the following three topics : an exact notion of the equality sign (=), the part-whole relation, and the action of controlling the outcome of an arithmetic operation. The results of the experiment will be discussed.

1. Introduction

One of the major developments in recent American research on cognitive processes has been in the analysis of problem solving (see e.g., Newell & Simon, 1972; Greeno, 1978b). In this research, important changes have occurred in the methodology as well as in the content of the investigations. Research techniques are being developed that try to analyze the internal mental processes underlying performance. As to the content of the studies, it is obvious that the tasks being studied are now much more representative of real life and school-like problems (Greeno, 1978a; Hayes & Flower, in press). A consequence of this development is that American research comes closer to the type of studies that have been practiced in the Soviet Union for several decades - namely, investigations in real school settings oriented toward analyzing and improving the actions underlying the learners' performance (Kalmykova, 1970). This approach has influenced research on problem solving in the Netherlands and in Belgium during the seventies (De Corte, 1977). For Western Europe, the approach is not entirely new but is clearly in line with contributions to the field that derive from the work of Selz (1935), Duncker (1935), Kohnstamm (1952) and De Groot (1946).

In this paper we wish to present an example of this approach. The example relates to children's solution processes in elementary arithmetic problems such as the missing addend problem. The choice of this kind of problem as an object of study was inspired by contacts with elementary school teachers. Related work has been done in the United States (Groen & Parkman, 1972; Groen & Poll, 1973; Rosenthal & Resnick, 1974; Heller & Greeno, 1978; Riley & Greeno, 1978; Lindvall & Ibarra, 1978), in the Soviet Union (Atachanov, 1969; Mikulina, 1969), and also in the Netherlands (Van Eerde & Verhoef, 1978; Wolters, 1978).

Two objectives were set forth when we started this study. The main objective was to make a contribution to the solution of several central problems in instructional psychology. More specifically, we had the following questions in mind : which phases can be distinguished in processes of problem solving; which research methods are valid, reliable, and efficient in collecting data on the internal cognitive processes during problem solving; can children's problem-solving ability be improved through instruction; and, if problem solving can be learned, what is the relation of such learning to cognitive development ? These questions can be approached from different domains of subject matter. We have chosen elementary numerical addition and subtraction problems as a subject of study.

The second objective is more practical in nature. A conclusion from our discussions with teachers was that the learning and teaching of indirect arithmetic tasks, such as the missing addend task, pose a real problem in today's classrooms. We presumed that the shortcomings in current mathematics teaching are mainly responsible for this problem. Through this study we have hoped to make a contribution toward remedying this situation.

2. A conceptual framework for elementary addition and subtraction problems

Based on discussions with teachers and on previous research, we have developed a two-dimensional conceptual framework in which children's errors as well as their appropriate and inappropriate structures of action in connection with the elementary addition and subtraction problems can be classed. One dimension of the framework relates to the characteristics of the tasks; the second dimension has to do with the problem-solving process. To describe the universe of elementary addition and subtraction tasks appropriately, we have retained four structural characteristics of those problems :

(a) The mode of presentation, meaning numerical versus word problems, the former of which has been emphasized in the present study with the exception of a few word problems in the posttest;
(b) The nature of the arithmetic operation - addition versus subtraction;
(c) The complexity of the problem - simple ($a \pm b = c$) or complex tasks; and
(d) The place of the unknown in the problem, discriminating between direct versus indirect tasks in which an addition is direct when the sum is unknown and indirect when one of the addends is unknown, and in which a subtraction is direct when the difference is unknown and indirect when either the minuend or the subtrahend is unknown.

With those four characteristics, we can describe each elementary problem : for example, .+4=9 is a simple, indirect, numerical addition problem, while 16-4-8=. is a complex, direct, numerical subtraction.

The second dimension of the framework consists of the possible stages or phases that we have distinguished in the problem-solving process - namely, a thinking phase, a technical or executive phase, and a control phase.

The thinking phase involves all the pupil's actions with respect to the tasks that lead to the decision to carry out a specified operation. Once this phase of problem analysis and/or problem transformation is finished, the corresponding operation is actually carried out; that is called the technical phase.

Finally, after a solution is obtained, the pupil can carry out identical or alternative operations on the task material to verify the correctness of his solution; these control actions constitute the control phase.
The three phases described cannot be distinguished or indicated in every solution process in a obvious way. It often occurs that actions within one or more of those stages are performed only partially or in a very abbreviated form. It is even possible that a pupil gives the solution to an arithmetic task without any trace of a thinking, a technical, or a control phase - namely, when he or she knows the answer by heart.

3. Overall research design

The overall design of the investigation has been inspired by the distinction made in the Russian instructional psychology between ascertaining and teaching experiments (Kalmykova, 1970). According to Kalmykova (1970, p. 128), an ascertaining experiment is used with the aim

> "of determining the peculiaralities of assimilation and application of knowledge, of the formation of habits, and of the modes of intellectual activity under given, already formed, conditions of learning".

In the ascertaining part of the study, arithmetic tests were collectively administered to 176 boys and girls of three first and four second grade classes. Each test consisted of 24 addition and subtraction items. We give a few examples to illustrate the different item types : .=3+7; .-7=5; 2+5+9=.; and .-4=6-2. In two of the seven classes, two tests were administered. In all, we obtained 233 answer sheets that were analyzed quantitatively and qualitatively.
To get a better understanding of children's solution processes and of their errors and difficulties with these types of arithmetic tasks, it is not sufficient to look at their performance. Therefore, a qualitative psychological analysis of the mental actions or processes (e.g., the problem-solving methods and strategies applied) underlying performance is necessary. The qualitative study of the answers on the collective tests was a first contribution in that direction. However, to collect more detailed data about these actions, one of the tests of twenty four items was administered individually to 12 children; 6 of them were bright and 6 were weak in arithmetic. These children were asked to think aloud while solving the items or, if that seemed to be difficult, to report retrospectively about the solution steps and the related considerations they had made. Occasionally, additional data were gathered through other techniques: the observation of the child's behavior, measuring response latencies, and asking the child for his opinion about the level of difficulty of an item.

On the basis of the knowledge obtained and the experience acquired during the ascertaining experiment, we have then designed a systematic teaching experiment. Contrasting both types of investigation, Kalmykova (1970, p. 132) gives the following definition of a teaching experiment :

> "If in the ascertaining experiment the researcher registers the results of an already formed method of instruction, then in the teaching experiment he himself constructs these methods in accordance with a hypothesis, outlined on the basis of observation and preliminary experiments, about the influence of one or another fact on the process of instruction."

In the teaching experiment we developed an experimental teaching program consisting of five teaching-learning units. The objective of the program was to remedy children's errors and difficulties established in the ascertaining experiment by equiping them with relevant and appropriate knowledge and problem-solving methods. The program was implemented by one of the researchers in a second grade class during a two week period in the beginning of the school year. The efficiency of the program and its underlying hypothesis concerning problem-solving processes were tested according to the pretest-posttest design with control group (Campbell & Stanley, 1963).

4. Quantitative analysis of the data of the ascertaining study

A quantitative error analysis was carried out on the data of the collective tests administered to 176 first and second graders. The objective was to locate the "afflicted areas" within the conceptual framework described above. In all, we found 1633 errors. First, we have examined whether those errors were primarily due to the thinking or to the technical phase of the solution process. Table 1 gives an overview of the nature of the errors.

Insert Table 1 about here

As shown in Table 1, the number of errors due to choosing the wrong arithmetic operation (as in $\underline{20}=12-8$; $17=\underline{30}+13$; or $7=13-\underline{20}$) is the sextuple of the number of technical errors (such as $\underline{5}=12-8$; $17=\underline{14}+13$; or $7=13-\underline{8}$). From these findings we conclude that for those children - who had all had arithmetic instruction for at least one year - the difficulties with respect to the elementary addition and subtraction problems are set mainly in the thinking phase of the solution process.

The classification of the errors as thinking or technical was done by the investigators. However, for a random sample of 10 errors, the classification was checked for objectivity. Sixteen student teachers who were informed about the meaning of the concepts of thinking and technical error, agreed almost unanimously with the investigators' classification for 9 of the 10 errors. Disagreement over the remaining error (17=13+17) was due to the student teachers' not understanding how it came about.

Second, we examined on which types of numerical problems the children committed the most errors. We mention here only the most noteworthy findings.

The percentage of errors was very high for some indirect, complex problems - especially subtractions. For example, the problems 20-.-12=4 and .-4-5=3 were answered correctly by less than 50 % of the pupils; the problem 6-2=.-4 yielded only 20 % correct solutions.

A striking result is that the direct and the indirect simple tasks were solved equally well with 87 % correct answers given for both of them. This result is striking because it is not in line with the current opinion in educational practice that indirect problems are much more difficult than direct ones. However, to get a differentiated picture and an appropriate interpretation of this finding, it is necessary to consider further distinctions within both the direct and the indirect simple tasks. Therefore, the following types or problems have been distinguished :

(a) Direct problem, Type I : $a \pm b = .$;
(b) Direct problem, Type II : $. = a \pm b$;
(c) Indirect problem, Type III : $a \pm . = b$; and
(d) Indirect problem, Type IV : $. \pm a = b$.

Table 2 gives the percentage of correct solutions for those four types of tasks separately for addition and subtraction.

Insert table 2 about here

From Table 2 it can be derived that the absence of difference in the percentage of correct solutions between direct and indirect simple tasks can, to an important degree, be accounted for by the Type II problems (.=a±b). The teachers of the classes where our tests were administered were very much surprised by the many errors on these tasks. It seems that those problems are very unusual. We have not found one arithmetic manual or research report in which they are mentioned.

A second point shown in Table 2 is that the major part of the errors on indirect problems were committed on Type IV subtractions. Case (1977, p. 3) has also pointed to the difference in number of errors between Type III and Type IV subtractions.

Finally, we mention that indirect problems were solved about equally well when the equality sign was placed in front (b=a+. instead of a+.=b). Therefore, we have not made a further distinction within the indirect problems. It can be remarked that this finding is in contrast with results obtained in the U.S. by Lindvall & Ibarra (1978).

5. Toward a qualitative, process-oriented analysis

In the preceding section we have located the so-called "afflicted areas" within the conceptual framework described before. The first and second graders whom we have examined seem to have difficulties mainly with respect to the thinking phase on indirect, complex subtractions and on simple Type II problems and Type IV subtractions. Those findings, interesting as they are, do, however, not provide us with sufficient insight into children's solution processes with respect to elementary addition and subtraction problems. Therefore, a qualitative action-oriented analysis is necessary. In the present investigation, this was carried out through an in-depth qualitative error analysis of the collective tests and on the basis of analysis of the thinking-aloud or retrospection protocols of the individual sessions with 12 children.

It is not possible to treat here the qualitative data relating to all the cells from our conceptual framework. We will limit ourselves to an exemplary discussion of the two "afflicted areas" with respect to simple addition and subtraction problems that were identified in Table 2 - namely, the thinking phase in connection with direct Type II problems and indirect Type IV subtractions. In the discussion of both types of problems, attention will be given to appropriate as well as to inappropriate structures of action in view of the solution of a problem. Although we are concentrating on the thinking phase of the solution process, we will, nevertheless, make a distinction within our analysis of appropriate and inappropriate structures of action between two groups of pupils : on the one hand those for whom the task is a real problem and, therefore, requires thinking, and on the other hand, those who consider the task as being of a familiar type for which they have an answer or a ready-made solution strategy immediately available, so that there is almost no thinking phase. In this connection, it should be obvious that whether a task represents or does not represent a real problem for a pupil has no immediate relationship with giving or not giving a correct solution.

6. Qualitative analysis of the thinking phase on Type II tasks (.=a±b)

6.1. Appropriate structures of action

Most pupils who answered the Type II tasks correctly did not show a thinking phase while solving the addition or subtraction. These tasks did not confront them with a real problem; immediately after reading the task, they knew a ready-made solution strategy - namely, adding in case of a plus sign and subtracting in case of a minus sign. When we asked Tom how he could give the solution for .=3+7 so quickly, he said, "I just know it. Tasks like that one are solved by adding the two numbers !"

For other pupils, these tasks were totally unfamiliar. One reason for that is certainly that such tasks occur rarely, if ever, in most of our instruction manuals. These children found a solution strategy only after a relatively long time or after having established through a control action that a given answer was not correct. The task was a real problem for these pupils, and they contrived to perform an appropriate arithmetic operation only after applying intentionally an important property of the equality relation - its reversibility. Through this reversion there appears a familair task. As an illustration we mention Kurt's thinking-aloud process for the task, .=3+7 : ".... We have not yet learned that ... Or, wait a minute; that is the same as 3+7=., because I may change places."

6.2. Inappropriate structures of action

We have seen in Table 2 that surprisingly many errors were committed on the Type II tasks. From the data of the individual sessions it was apparent that the pupils who gave no or wrong answers can be divided into two groups.

First, there are the weak pupils for whom the Type II tasks represented unsolvable problems. They fail to transform these unfamiliarities to Type I items and did not come to a decision to perform an arithmetic operation. A plausible interpretation for this finding is that these children had very deficient knowledge of the equality concept; they had not acquired transferable or maneuverable structures of action relating to this concept. For example, one pupil who failed on all Type II items gave the following definition of the equality sign (=) :
"the sign behind which you have to write the outcome of the task".
Most pupils in this group did not answer the Type II problems or told us that they had not yet learned to solve such tasks.

The second group of pupils did not experience the Type II items as real problems. As a consequence, there was no thinking phase in their solution process. They gave a wrong answer either immediately after reading the task or after a technical phase. In line with some findings of Brown & Burton (1978), we discovered that their errors were due to a very systematic procedure. Nineteen percent of all pupils answered the item .=3+7 with 4. The items .=4+6 and .=12-8 were often answered with 2 and 20 respectively. During the individual sessions, we found that the children who committed such errors did not even notice the peculiar structure of the items. They solved those tasks as if they had another structure that was familair - namely, .+3=7; .+4=6; and .-12=8. In other words, these pupils solved the items that they expected, not the ones that were actually given. Some present practices in arithmetic teaching are probably responsible for that state of affairs. Children are often given stereotyped series of arithmetic tasks, and the more or less implicit evaluation criterion is the number of items solved. This has two harmful consequences : (1) the thinking phase involving problem analysis fails to turn up, and (2) the control phase to check the solution obtained is omitted.

7. Qualitative analysis of the thinking phase on Type IV subtractions (.-a=b)

7.1. Appropriate structures of action

The Type IV subtractions occupy a special place among the indirect simple addition and subtraction problems. We can distinguish here four kinds of tasks :

- Type III additions : a+.=b;
- Type III subtractions : a-.=b;
- Type IV additions : .+a=b; and
- Type IV subtractions : .-a=b.

The solution method that most pupils apply without difficulties to Type III additions and subtractions and to Type IV additions cannot be used to solve Type IV subtractions. For indirect additions this method is as follows : the counter is set to a and is then incremented until its value is equal to b; the solution is the number of times the counter has been incremented (Groen & Poll, 1973). For Type III subtractions the counter is set to a and then decremented until its value is equal to b; the solution is the number of times the counter has been decremented. This procedure is not applicable to Type IV subtractions because there is no basic number from which the counting off can start

We will now discuss the kinds of actions that occurred in children who answered the Type IV subtractions correctly. First, we report two solution strategies of the pupils for whom the tasks represented a real prolem; these procedures were ascertained mainly in bright pupils.

We illustrate the first solution procedure by means of a thinking-aloud protocol of seven year old Hans, who was completely unfamiliar with the Type IV subtraction. When he was confronted with the item .-7=5, he said, "Whew, I can't do that one, because we have not learned it yet !" When we pressed him on an answer, he continued, "Wait, I will try a number. That number must certainly be greater than 7. When I use 9, the outcome is 2. That is too little. 10 minus 7 equals 3... yah, that is 2 too little. Thus, the answer must be 12. 12-7=5; O.K. !" This strategy starts with a stage of problem identification : the pupil realizes that he cannot find the answer by just subtracting the two numbers given. The procedure that is then applied is rather circuitous and makes high demands upon the child's working memory, certainly when the technical aspects of subtracting are not very well mastered. We call this strategy "intelligent trial-and-error".

In the second procedure, the pupil solves the Type IV subtraction (.-a=b) by performing the operation a+b. It is important to know how a child, for whom the task is a real problem comes to the decision to perform this operation. For,one should realize that those first and second graders are not familiar with elementary algebraic principles for solving equations; therefore, it is very unlikely that they will transform a Type IV subtraction to a direct addition by means of such syntactic principles. In our data we have found two different patterns of actions.

On the one hand, there is a semantic approach that is shown by pupils who try to transform the numbers and symbols of the task into a semantic, cognitive schema that they have available and in which addition and subtraction as inverse operations are integrated. An example of such a semantic solution procedure was found in an intelligent first grader. Confronted with the item .-7=5, he said, "I take 7 away from a certain number, and 5 remains. To know how much that number was, I must add to 5 the 7 that I have removed before." As a further illustration of this semantic procedure, we mention also the solution strategy of a pupil who approached the Type III addition 4+.=9 with the concepts of part and whole in saying, "I have one part, 4; there is another part, and together they are 9. When I decrease 9 by 4, I know how much the other part is." Before describing the second thinking process that gives rise to performing the inverse operation to solve a Type IV subtraction, we would like to give a comment on the preceding very intelligent mental actions. The two examples described can be related to two of the three semantic schemata for representing arithmetic problems that Greeno and others (Heller &

Greeno, 1978a; Riley & Greeno, 1978) have distinguished in their work on word problems - namely, the cause/change schema in the first and the combination schema in the second example. In other words, those two pupils have spontaneously constructed and applied a semantic schema to solve unfamiliar numerical problems, for which they had no ready-made solution available. Data of this nature might lead to questioning the traditional view concerning the relation between numerical and word problems. In this traditional view, word problems are usually considered as being more difficult than numerical tasks. The preceding data suggest rather the opposite - namely that the semantic content of word problems, if clearly stated, might support and facilitate the solution process of unfamiliar problems, especially in young children. Support for this view can also be found in results obtained by Lindvall & Ibarra (1978, p. 5). In their study only 25 % of the investigated primary grade pupils solved numerical tasks of the type .-a=b correctly, while word problems with the same basic structure were solved correctly by 56 % of the pupils.

On the other hand, we observed an interesting development in the actions of Hans after he had solved the item .-7=5 by means of intelligent trial-and-error as described above. He discovered the second procedure (perform a+b); however, he had no insight into it and was not able to justify the procedure. During the individual session with Hans, several items of another type followed after .-7=5. Then, he was presented a Type IV subtraction (.-6=5) again. Initially, he was once again at a loss of what to do.

Suddenly, he remembered that he had already solved a similar item and started to look for it. He found the item 12-7=5 but failed to remember how he had solved it. Suddenly he said, "Perhaps I have added 7 and 5, because that equals 12. The outcome of this problem should then be 6+5=11. That fits; for 11 minus 6 equals 5. Now I have found a trick to solve such problems. You only have to add those numbers."

Until now we have discussed the appropriate structures of action of those pupils for whom the Type IV subtractions were a real problem. However, for most children who answered this type of tasks correctly, there was no trace of a thinking phase in their solution process. They started to perform an arithmetic operation almost immediately after presentation of the task. Afterwards, they told us that they knew how to solve such subtractions. In other words, for these children, the Type IV subtractions were familiar tasks for which they had a ready-made solution strategy available. Their solution process can be described as follows. On the basis of external characteristics, the item is identified as belonging to a certain type. Consequently, the corresponding solution procedure ("add the two give numbers") is retrieved from memory and applied to the item. A lot of those pupils performed this procedure blindly or by rote, without being conscious of what they were exactly doing. The justification was often as follows : "the teacher has told us to do it in this way." This contrasts strongly with the solution process of other children who also came up with the answer quickly without a preceding thinking phase but were able to justify their procedure.

7.2. Inappropriate structures of action

From the quantitative analysis, we know that a lot of errors were committed on Type IV subtractions. Children for whom the tasks were a real problem and those who perceived the tasks as being of a familiar type made the same mistakes. We give some examples : 5-6=11; 2-7=5; 10=9-1; and 4=5-9. The error can be easily reconstructed : the greatest number is decreased by the smallest one and the difference is written down as the solution. However, the interesting question is also how children come to this procedure and how they justify it. The thinking-aloud protocols revealed three sets of actions.

Some children who committed errors like 5-6=11 and 2-7=5 (but also 17=30+13 and 3+12=9) simply transferred the familiar and easy solution method for direct simple tasks to indirect problems. So, the Type IV subtractions are solved in the same manner as the Type I subtractions by decreasing the greatest number by the smallest one. A first grader justified his answer for the item 2-7=5 as follows : "You see that there is a minus sign ! Then you have to subtract, don't you ?"

Other pupils distinguished between direct and indirect tasks, but they confused the solution methods for the different indirect simple items. The cause of this confusion is probably that the relations between task types and solution procedures have not been acquired gradually and with insight. We found an extraordinary example in a second grader who gave the following solutions : 7-8=1 and 12-17=5. Referring to subtracting in case of a Type IV task and to adding in case of a Type III task, he gave the following justification : "The teacher has told us that you have to solve such problems in this way." He guessed but was wrong !

A third type of errors occurred only on Type IV subtractions with the following structure : a=.-b in which b > a. Such items were often answered incorrectly because the pupil read, represented and solved the task from the right to the left, urged to do so by the place of the equality sign. This is illustrated by a thinking-aloud protocol of a second grader relating to the task 4=.-9 : "Nine minus what equals 4 ..., if I decrease 9 by 4, then I know the number that has to be filled in."

8. In search of an interpretation

Having described children's actions that lead to correct as well as incorrect answers on Type II tasks and Type IV subtractions, we now go in search of the underlying causes of the errors on those types of arithmetic problems,

especially on the indirect ones. In the literature, we have found three general interpretations of children's difficulties with respect to the thinking phase in solving indirect arithmetic problems.

First, there is the Piagetian point of view. According to this conception, children who commit a lot of errors on indirect tasks have not yet acquired the operation of reversibility, and this is due to their not having reached the necessary level of cognitive development. This explanation does not satisfy us for two reasons. On the one hand, it can be noted that, as we have seen in 7.1., indirect tasks can also be solved in a way that does not require the application of the so-called inverse operation. On the other hand, we have at present a sufficient amount of research evidence showing that, through development-stimulating instruction, children can manage tasks that are unsolvable at their age according to Piaget's theory. Experiments by the Russian Gal'perin and others are illustrative in this respect (De Corte, 1977).

A second interpretation is suggested by researchers in the field of learning disabilities. They explain the numerous errors on indirect problems in terms of disturbances in the young problem solvers.

For example, Borghouts-Van Erp (1976) states that some children are not capable of grasping the connection between the elements of an arithmetic task and are, therefore, entirely committed to arbitrary and superficial associations; these children are disturbed at the level of thinking. There are again two reasons why we are not satisfied with this explanation. When so many pupils make errors on indirect tasks, it is necessary, then, to take up primarily and directly the teaching-learning situation as a means to prevent disabilities. A second source of dissatisfaction is that the approach of Borghouts-Van Erp lacks a dynamic, process-oriented approach to cognitive functioning.

We agree with the third interpretation of children's errors on indirect problems but also on direct Type II tasks. According to this explanation, the errors are not in the first place due to children's level of cognitive development or to learning disabilities but primarily to shortcomings in instruction (Atachanov, 1970). In this respect, a major shortcoming of instruction is that children often acquire only specific and isolated methods for solving problems and tasks. As a consequence, they lack an appropriate orientation basis for approaching more or less new and unfamiliar situations and problems. The concept orientation basis that is central to the learning theory of Gal'perin, means the totality of conditions upon which the learner orients himself in performing an action. For optimal learning, the learner must, according to Gal'perin, have a complete orientation basis; this means he must possess all the conditions necessary to perform an action correctly from the beginning (Gal'perin, 1969, 1970; De Corte, 1977). If children would have a complete orientation basis with respect to elementary addition and subtraction items, they would be capable of approaching all tasks intelligently and systemati-

cally, also the unfamiliar ones for which they do not immediately have a ready-made solution procedure. This brings us to the important question of how children can be provided with an appropriate orientation basis. According to our view of problem solving (De Corte, 1980), this would involve children's being equipped with two complementary kinds of structures of action : (1) structures of action that consist of being able to use and apply relevant conceptual knowledge of subject-matter content such as concepts, principles, etc.; and (2) thinking procedures and techniques for handling, analyzing, and transforming problem situations so that they make contact with specific subject-matter content. These two kinds of structures of action are related to the distinction that Greeno (1978a, p. 17) has made between the mechanics of problem solving and the semantics of problem solving.

With respect to thinking procedures, pupils should acquire the attitude and the skills to analyze and represent the relations between the data of the problem before starting to perform computations. Indeed, we established that, when reading of the task did not immediately remind pupils of a possible solution method, only a few of them came to apply techniques for problem analysis such as constructing a schema of the problem or verbalizing the written symbols. Besides techniques for problem analysis, control actions form another important component of the equipment of an efficient problem solver. It was very rare that we found a child who performed consciously and with insight a control action to verify the correctness of his answer to a task.

With respect to conceptual knowledge, we point to the importance of having available a set of clear and powerful mathematical concepts and relations that can be used during problem analysis and transformation. In connection with elementary addition and subtraction problems, relevant conceptual knowledge certainly involves the equality concept and the part-whole relation. To be able to solve arithmetic tasks with insight, it is necessary to possess a clear and complete understanding of the equality sign. The many errors that we have found on Type II tasks (.=a±b) furnish obvious proof that this notion is not or is not sufficiently mastered by a lot of pupils. Complementary to the illustrations given earlier (see 6.2.), we mention one noteworthy error that can be interpreted in terms of a deficient equality concept. Twenty percent of the pupils solved the complex problem 9-.=8+. as follows : 9-1=8+1. They performed two separate operations : 9-1=8 and 8+1=9. They then wrote down the two outcomes in the open spaces of the complex task. Our data show that a lot of pupils hardly realize that the equality sign implies a judgment concerning the quantitative equivalence of the left and the right terms of the task. It seems as if those children consider the unknown number in a given task only as the result of one or more operations, without being

explicitly aware of the fact that an unsolved arithmetic problem involves the representation of quantitative relations and proportions. It is precisely in this respect that the equality sign fulfils an essential function.

Besides a clear equality concept that can be helpful during the thinking phase of the solution process, a lot of pupils also lack an appropriate conceptual schema with which they could represent the relations between the given and the unknown elements of an elementary addition or subtraction problem. It seemed to us that the part-whole relation is, in this respect, a useful and adequate aid to represent the problem data and to infer from it the appropriate arithmetic operations. This idea was derived from the literature, especially from some Russian and Dutch work (Atachanov, 1969; Mikulina, 1969; Wolters, 1978), on the one hand, and from our observations during the ascertaining study on the other; indeed we saw several bright pupils apply the part-whole relation spontaneously in their solution processes for elementary arithmetic problems. We will discuss the use of the part-whole relation more extensively in the next section. Here we only mention that this conceptual schema should enable the pupil to identify the given and the unknown elements of the task as parts and as the whole, and to determine each element through performing the appropriate operation on the others.
In summary, it is our view that the traditional methods of arithmetic instruction are primarily responsible for the many errors, especially on Type II tasks and on Type IV subtractions. Through these traditional methods, children have acquired a set of specific isolated solution methods that are almost not transferable to new and unfamiliar tasks and situations. To give children an appropriate orientation basis for approaching such new problems, it is necessary, first, to impart to them general and powerful concepts and principles such as the equality concept and the part-whole relation schema and, second, to teach them to apply these concepts and principles during a thinking phase that involves problem analysis and precedes the technical phase of performing an arithmetic operation and also during a control phase that follows the performance of the operation. In the following sections we will elaborate and test this hypothesis which constitutes the starting point of the systematic teaching experiment.

9. Objectives and design of the teaching experiment

The objectives of the teaching experiment have already been stated earlier in this paper. Nevertheless, it seems useful to repeat them here concisely. First, we wished to make a contribution to the testing of our view of problem-solving processes described in the preceding section. If instruction in the

hypothetical structures of action underlying problem solving yields success, in the sense that this instruction leads to the improvement of children's problem-solving ability, this is considered to be evidence supporting the hypothesized processes. Second, in the posttests used to establish the effects of the experimental teaching program, we have also inserted transfer items and a retention test. Thus, our results can be related to the problem of the relationship between learning and cognitive development. If children achieve success in solving transfer and retention items, this provides evidence for their acquiring maneuverable and stable cognitive strucures of action. According to Vygotsky (1963, p. 26-27; 1978, p. 83-84), this can be considered to be a step forward in cognitive development. Finally, we had a more practical objective - that of making a contribution to the improvement of instructional practice in connection with elementary addition and subtraction problems.

The systematic teaching experiment was set up according to the pretest-posttest design with control group (Campbell & Stanley, 1963).

A collective arithmetic test, constructed especially for this study was administered to two second grade classes which did not participate in the ascertaining study : an experimental class of 27 pupils and a control group of 25 children. This pretest made it possible to verify the results of the ascertaining study and yielded data about the initial level of arithmetic competence of both groups before the teaching program was applied to the experimental group. Table 3 gives an overview of the collective pretest.

Insert Table 3 about here

To six pupils of the experimental class (two high, two average, and two low ability children according to the teacher's opinion) and to three pupils of the control class (one of each ability level), a ten item qualitative test was administered individually. During this test the focus was again oriented toward the structures of action underlying performance. The first seven items represent rather unusual tasks, two of them even being unsolvable. They examine to what degree a pupil knows and understands a mathematical concept, relation, or principle and can apply this knowledge appropriately in analyzing problem situations. The last three items are taken over from the collective pretest; they were administered to observe children's problem-solving actions during the solution process. Table 4 gives an overview of the individual test.

Insert Table 4 about here

After the testing phase, the experimental teaching program, which will be described in more detail in the next section, was implemented by one of the investigators in the experimental class. Over a period of two weeks, he gave arithmetic instruction daily, covering eleven lesson periods in all. Meanwhile, the control class was instru[illegible] under the usual teaching program. During that period of the school year, this program mainly involved a review of the subject matter of the first grade where addition and subtraction with numbers less than 20 occupied a central place.

When the experimental teaching program was terminated, the posttest was administered collectively to the pupils of the experimental and control groups. Some items of the test were very similar to the tasks that occurred during the preceding instruction in both classes; other test items differed more or less from the problems that were treated during instruction. The same pupils as during the pretest phase were also tested individually again. One month later a ten item retention test was administered to the children of the experimental group. For practical reasons, it was impossible to implicate the control group in the retention phase.

10. Description of the experimental teaching program

The content of the teaching program was spread over five teaching-learning units (TLU). The first unit was devoted to the equality concept and comprised three lesson periods. Three units were devoted to the part-whole relation : a general orientation to the part-whole relation (two lesson periods), learning to handle the relation (three lesson periods), and solving numerical problems by means of the relation (two lesson periods). During the last unit, which took only one lesson period, control actions for elementary arithmetic problems were treated.

During the experimental teaching program, the pupils received 13 worksheets. These sheets carried out a double function : they were used to practice concepts and methods, and they provided us with data concerning the children's progress.

Although it is probably obvious from the preceding sections, we wish to state explicitly that, during experimental instruction, no special attention was paid to the technical aspects of performing elementary additions and subtractions. In other words, the focus was oriented to the thinking aspects of solving such problems.

10.1. First TLU : the equality concept (3 lesson periods)

In section 8 we have argued that a clear and complete understanding of the equality sign is a necessary condition for solving simple as well as complex direct and indirect problems with insight. This implies the following : (1) the pupil realizes that what is written on one side of the sign equals what is written on the other; (2) the pupil uses this knowledge at the beginning and at the end of his solution process for arithmetic problems; and (3) the pupil knows, understands, and applies in his solution actions the properties that derive directly or indirectly from the equality notion - for example, the interchangeability of the two terms of an addition or subtraction.

During the first lesson, children learned to use the symbols $=, \neq, >$, and $<$, in comparing continuous quantities (material objects as well as graphic representations) on different parameters such as length and content. The conception of this lesson was inspired by the work of Davydov (1972). In the second lesson, the pupils had to handle the same symbols to compare discontinuous quantities which were materially given or graphically represented. In the third lesson, children were confronted with unequal terms; they had to undo the inequality either by decreasing the greatest quantity or by increasing the smallest one. With graphic items (e.g., two umbrellas on one side and five on the other), this did not present difficulties for the children (they drew three additional umbrellas or crossed three of them out on the other side). However, items with numbers gave problems with respect to the notation conventions, albeit that the pupils performed the correct mental action. For example, on the item, "Make equal by adding or subtracting something : $2 = 4$," answers like the following were given : $\underline{+2}\ 2 = 4$; $2 = \underline{-2}\ 4$, and especially $2 = \underline{2} - 4$. Consequently, special attention was paid to the notation rules.

10.2. Second TLU : general orientation to the part-whole relation (2 lesson periods)

The part-whole relation is a conceptual schema that can help children represent the elements of a task together and in their exact relations to one another. Acquisition of this schema in a functional way increases the probability that children will not approach a task thoughtlessly anymore, but will try to make explicit the relations between the given and the unknown elements of a problem.

During the second TLU, the pupils made acquaintance with the part-whole schema, and they learned to handle the concepts part and whole to represent

the relation between continuous and between discontinuous quantities. The materials used in this unit were mainly real objects and graphic representations of objects. It was only at the end of the second lesson that we introduced some tasks with numbers.
We wish to mention the most important problems and shortcomings that we established in children's conception of the part-whole relation. A number of pupils initially had difficulty in grasping the relatively of the concepts; they did not understand that the same object can once be a part and then a whole. Other children did not distinguish in the beginning between the mathematical concept whole and the every-day word full. A third problem was that pupils sometimes named parts without the sum of these parts being the whole.

10.3. Third TLU : learning to handle the part-whole schema (3 lesson periods)

The objective of this TLU was for pupils to acquire two skills relating to the part-whole schema which are required to solve elementary arithmetic problems heuristically and with insight. These skills are as follows : (1) being able to identify the givens and the unknown in a numerical task as part or as whole, and to construct the appropriate schematic representation; and (2) being able to represent the value of each element of the schema by means of the other elements. The representation of the part-whole relation that we used is shown in Figure 1.

Insert Figure 1 about here

In the first and second lessons, the pupils learned to identify the parts and the whole in a numerical problem. On the basis of previous work (Assink & Verloop, 1977) and of our own experience, we thought that it was better not to use numerical problems in the beginning of this unit. The main reason for this is that, when confronted with a numerical problem, children start to perform an arithmetic operation very quickly without considering the mathematical structure of the task. Therefore, we have initially used exercises like those proposed by Case (1978) for teaching the missing addend problem. Children were given a series of tasks like the one in Figure 2, and they were asked to complete the drawing and to fill in under each element whether it is a part or the whole.

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Insert Figure 2 about here

Such exercises are worthwhile; by means of simple and suggestive material, they help the children to understand the logically and mathematically necessary place of the parts and the whole in a numerical problem in a gradual way.

In the third lesson, the children learned to represent an element of the part-whole relation in terms of arithmetic operations on the other elements. After the pupils were made familiar with the schematic representation of the part-whole relation (see Figure 1), a series of such schemata were drawn on the board. Children were asked to represent each element (the parts and the whole) by means of the other elements. An example is shown in Figure 3.

Insert Figure 3 about here

10.4. Fourth TLU : solving numerical problems by means of the part-whole relation (2 lesson periods)

For the first time in the course of the experimental teaching program, the pupils were now provided with a series of numerical addition and subtraction problems. One example was solved for the whole class; thereafter, the children were given twelve simple tasks. They were asked to solve each problem in three steps : (1) identify the parts and whole, construct the schematic representation, and note the givens and the unknown on the schema; (2) choose an arithmetic operation to determine the unknown; and (3) perform this operation and write down the outcome on the schematic representation as well as in the numerical task itself.

Only 5 of the 27 pupils committed one or more errors on the twelve problems. The second lesson of this TLU started with a discussion of these errors and the underlying inappropriate structures of action. This discussion was further oriented toward the formation of alphabetic formulae and schematic representations for complex additions and subtractions - for example, P + P + P = W and W - P - P = P. Finally, six complex numerical problems were presented. Due to a shortage of time, most pupils solved only two of them.

10.5. Fifth TLU : control actions for elementary arithmetic problems (1 lesson period)

As we have stated before, it seems to us that control actions form an important component in the equipment of a successful problem solver.

Therefore, we thought that teaching children such actions in connection with elementary arithmetic problems would also contribute to a reduction in errors. The design of this TLU started from the conception that mastering control actions implies the following : (1) the pupil knows which steps he has to take successively in order to verify the answer to a task; (2) the pupil can justify these actions in terms of meaningfulness and efficiency; and (3) the pupil applies the control actions spontaneously as a final step in his solution process to a problem.

In the beginning of the lesson, the control actions for indirect problems were imparted to the children by means of four simple tasks written on the board. These actions consisted of the following steps : after writing down the provisional answer, the indirect task solved is transformed to a direct one ($a \pm b = .$) by hiding the sum or the difference; the familiar Type I task is then solved, and the outcome is compared to the hidden number. Next, the children were given four indirect problems on which they had to apply the control actions. This lesson did not yield the effect that we had expected. A lot of pupils had not yet mastered the control actions very well after the lesson. However, this is not surprising. Indeed, one lesson period is certainly not enough to achieve this goal, not to mention the acquisition of a spontaneous control-oriented attitude.

11. Results of the experimental teaching program

In this section we will give an overview of the effects of the experimental program that we established through administering the collective posttest, the individual posttest, and the retention test.

11.1. Results on the collective test and the retention test

Table 5 shows the global results of the experimental and the control groups on the pretest, the posttest, and the retention test. The numbers indicate the percentage of correct answers. In this respect, we should remark that the scoring system for the posttest and the retention test was rather severe for the experimental class. An answer was scored as correct only when, in addition to the solution, the alphabetic formula (e.g., P = W - P) and the schematic representation of the part-whole relation were also correct.

Insert Table 5 about here

We can conclude from Table 5 that the experimental group made strong and stable progress, while this is not the case for the pupils of the control group who obtained the same results on the posttest as on the pretest. On the average, the children of the experimental class answered four items more correctly on the posttest as compared to the pretest. A more detailed analysis revealed that nine children of average ability made the greatest absolute progress, especially with respect to simple tasks - from 71 % to 94 %; a similar trend was observed for children of low ability - from 55 % to 75 %. We also examined the proportion between thinking errors and technical errors in the pretest and the posttest. As the experimental program was mainly oriented toward the thinking phase of the problem-solving process, we expected a strong decrease in thinking errors. The findings are shown in Table 6.

Insert Table 6 about here

The results are in line with our expectations. The experimental program yields a significant reduction in thinking errors in the experimental group. Items 18, 19 and 20 of the posttest (see Table 3) can be considered as transfer tasks in the sense that no items of this type were treated between the pretest and the posttest, neither in the experimental nor in the control group. As mentioned before, the retention test consisted of ten items that had the same level of difficulty as the collective posttest. Items 18, 19 and 20 of the posttest were solved correctly by, respectively, 6, 4, and 10 children of the control group (N=25); in the experimental group (N=27) we found, respectively, 16, 12 and 20 good answers (and correct schematic representations). As Table 5 shows, 79 % of the answers on the retention test were correct. These data suggest that the effects of the experimental program were not specific and shortterm but that the children had acquired structures of action that were relatively stable and also maneuverable toward problems of a different kind than the ones treated during instruction. According to Vygotsky's view referred to before (see section 9), these data could be considered to be an indication that the children made progress in their cognitive development. However, our present data are too rough and too scarce to warrant this conclusion. Nevertheless, they justify maintaining the hypothesis concerning the influence of learning on cognitive development for further research. On the other hand, it can be added that there is already a considerable amount of research supporting this hypothesis (see, e.g., De Corte, 1977).

11.2. Results on the individual test

The ten item qualitative diagnostic test that was administered individually to six pupils of the experimental and three of the control group (see Table 4), was intended to reveal children's structures of action underlying performance. More specifically, the test examined to what degree the children knew, understood, and could apply certain mathematical concepts and thinking techniques. To score the items quantitatively, we took three aspects into account : the answer, the solution procedure, and the justification of this procedure in terms of the relevant concepts and solution methods.

The quantitative results of the individual pretest and posttest are given in Table 7.

Insert Table 7 about here

The data in Table 7 show that the six children of the experimental group made significant progress from the pretest to the posttest, while this was not the case for the three children of the control group. The qualitative data revealed that the strong progress of the experimental group was not primarily due to an increase in the number of correct answers but to the fact that in the posttest more children used spontaneously appropriate concepts and methods and could give a satisfactory justification for their solution procedures. To illustrate this we mention two examples. The first item of the test was, "Correct or not : 8+4=4+8 ?" In general, pupils of the experimental group approached this task in a different way in the posttest than in the pretest and also in a different way from the pupils of the control group. They did not start to count immediately but analyzed the task and found the solution after a few seconds without performing an arithmetic operation. One pupil of the control group needed twenty seconds to reach the correct answer. She justified her solution as follows : "Eight plus four equals twelve, and four plus eight also equals twelve. Both outcomes are equal; so, 8+4=4+8." On the question whether this could also be true with other numbers, she did not dare to give an a priori answer.

For the unsolvable items, numbers 5 and 6, several children in both the experimental and the control classes anyhow gave a solution in the pretest phase : e.g., 7=5+1+3 and 7-1=8. In the posttest phase, however, almost all the experimental pupils pointed out the unsolvability of these tasks. About item 5 one said, "How is it possible ? The parts together are greater than

the whole !" A pupil of the control group, after having tried three possible answers for 7-.=8, decided also that the item was unsolvable. However, it took him a long time, and he could not justify his answer. The preceding findings allow us to maintain the hypothesis that the children of the experimental group acquired throughout the experimental program a heuristic procedure that consists of analyzing the problem before starting to count and to perform operations. During this stage of problem analysis, they made spontaneous use of the conceptual knowledge learned and were able to justify their actions in terms of this knowledge. Combined with the results of the collective tests, this would mean that the quantitative improvement of the performance of the experimental group on these tests can be interpreted in terms of a qualitative change in the children's problem-solving actions.

12. Summary or results and discussion

The results of the collective posttest show that the experimental teaching program applied in a second grade class has led to a strong decrease in children's thinking errors on elementary addition and subtraction problems. The analysis of the qualitative diagnostic data of the individual pretest and posttest suggests that the quantitative improvement in achievement is due to a qualitative development in children's problem-solving actions. The outcomes of the transfer and retention items give some indications that the effects of the experimental program are not specific and shortterm but that the children have acquired stable and maneuverable structures of action. For this last finding, we have only sparse data, but they are in line with ample research evidence supporting Vygotsky's view of the relation between learning and cognitive development.
In trying to reduce children's thinking errors, our experimental teaching program was not symptom-oriented, but we chose a more fundamental approach based on our view of the problem-solving process. This implied that children should be imparted relevant conceptual mathematical knowledge and equipped with heuristic techniques for problem analysis and with a set of control actions. This provides them with an appropriate orientation basis for approaching new problems and situations. Our findings point out that children's arithmetic action patterns can, indeed, be systematically influenced, and, in so doing, they support our view of problem-solving processes. It is at the same time obvious that this kind of research, in which an ascertaining and a systematic teaching experiment is combined, can contribute to the testing of hypotheses about problem-solving processes. In summary, the present study justifies continuation of this type of research from a theoretical as well as from a more applied standpoint.

Our approach to children's problem-solving processes for elementary arithmetic tasks was strongly influenced by the action-oriented approach described in the introduction to the present symposium.

In the first place, we were oriented toward the analysis of children's arithmetical actions during the solution of elementary addition and subtraction tasks. Taking into account time and space limitations, we have mainly discussed children's thinking actions; however, we also analyzed their structures of action during the technical phase of the solution process, where we found a considerable number of examples of automated and abbreviated actions.

Furthermore, the overall design of the study was obviously inspired by two related characteristics of Eastern European research in instructional psychology: a concern for external or ecological validity as well as for the integration of basic research oriented toward theory construction and applied research which places an accent on practical relevance. Indeed, our investigation can be characterized in Kalmykova's terms (1970) as a natural experiment with an ascertaining and a systematic teaching component. We think that this kind of research can contribute to the development and elaboration of a theory of children's problem-solving processes but can yield, at the same time, data that are relevant for instructional practice, more specifically in this case for instructional design.

Finally, the elaboration of the ascertaining as well as the systematic teaching experiment was in line with some other characteristics of the Eastern European approach - namely, qualitative diagnosis and developmental instruction. Our tests, which were constructed specially for this study, tried to obtain insight into the actions underlying children's performance and errors and, during the individual sessions, they took the form of an individual interview. The experimental program was explicitly intended to teach children stable and transferable structures of action that provide them with a complete orientation basis to approach, analyze, and solve new problems efficiently. In other words, we intended to realize that "by taking one step in learning a child moves two steps in development," (Vygotsky, 1963, p. 27). Our data in this regard are, as said before, only rough and sparse; however, they are convergent with the findings of other researchers. An additional point to be mentioned is that our eleboration of the experimental teaching program was inspired by a key principle of Russian instructional psychology derived also from Vygotsky's work : that powerful mental actions can best be acquired through the interiorization of external activity.

It is now interesting to establish that, on different points, our study bears a more or less strong resemblance to ideas and data found in recent American instructional psychology. Regarding the conception of research, there is also a trend in the U.S.A. toward the integration of basic and applied research (Glaser, 1978; Greeno, 1978c). Another important development is certainly that the teaching experiment is more and more recognized as a valid research strategy (Larkin, Heller & Greeno, 1979; Resnick & Glaser, 1976).

Furthermore, some of our views and findings can be related to data reported in recent American work. First of all, our conception of problem-oriented thinking, in which conceptual knowledge as well as problem-solving procedures occupy both a central and mutually related role, is convergent with the present view of American researchers such as Greeno (Greeno, 1978a; Heller & Greeno, 1978a). At a more detailed level, we have also referred to the relevance to the interpretation of some of our observations of Greeno's distinction between syntactic and semantic solution strategies (Heller & Greeno, 1978b; Riley & Greeno, 1978). Although we have not indicated so previously, we have found in our data, especially in those relating to the technical phase of the solution process, examples of solution procedures which support Resnick's idea concerning the role of invention in children's learning to solve arithmetic tasks (Resnick, 1976; Groen & Resnick, 1977). It seems to us that a lot of such inventions can be related to anothe notion that occupies a central position in Soviet theory about the learning of cognitive skills - namely, the concept of abbreviation. Referring to Russian studies, Van Parreren (1978) has described three forms of abbreviation spontaneously or autonomously occuring during learning processes. One of these forms of abbreviation that is relevant to invention is called "selecting the shortest route to the goal", which "rests on the subject's discovery of identity between two different paths to a goal and the selection of the shortest route to the goal," (Van Parreren, 1978, p. 9). As an example, the author refers to learning the operation of adding. Finally, as we have mentioned before, some of our data accord very well with findings reported by Brown & Burton (1978). The following quotation from their paper (Brown & Burton, 1978, p. 157) applies remarkably well to some of our observations :

> "A common assumption among teachers is that students do not follow procedures very well and that erratic behavior is the primary cause of a student's inability to perform each step correctly. Our experience has been that students are remarkably competent procedure followers but that they often follow the wrong procedures."

Earlier in this section, we have stated that the present study justifies continuation of this type of research. We would like to add now that such continuation is certainly necessary; indeed, we are very well aware of the limited scope of the findings of this more or less exploratory investigations, and,

therefore, we guard against hasty, undifferentiated and definitive conclusions. In the first place, the theoretical framework of this study was rather rough and general in nature. It is obvious that further theoretical work is now necessary, taking into account the results obtained together with related ideas and findings of other researchers to whom we have previously referred.
Secondly, our study is subjected to several methodological weaknesses - some of them being related to our concern for external validity : for example, the relatively short duration of the experimental teaching program, the small size and slight representativeness of our experimental and control groups, and the restriction that we were not able to collect detailed data about the teaching and learning activities in the control group during the teaching experiment. Besides taking into account those shortcomings in future studies, we also intend to refine and broaden our data-gathering techniques. Indeed, it is our opinion that it is necessary to use a series of techniques concurrently in order to obtain a good and complete picture of the cognitive processes and the mental actions that are taking place in the human mind. In this regard, the use of laboratory experiments in combination with natural experiments seems to us a valuable strategy.
Finally, a critical examination of our experimental teaching program has also revealed some weaknesses in its content with respect to the equality concept and the part-whole relation as well as with respect to the control actions. We mention two examples. The part-whole relation, as we have used it, has only limited value as a cognitive schema for analyzing and representing elementary addition and subtraction tasks. The schema cannot be applied to some problems - e.g., 4+7=10+. or 6-2=.-4. Furthermore, the time provided in our program for teaching the control actions as a heuristic device was far too short to attain the objectives intended. Our experience within the present study, however, has been very useful in the perspective of the design of related and new experimental teaching units.

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Table 1. Distribution (in %) of the 1633 errors on the collective tests according to their nature

Thinking errors	78 %
Technical errors	13 %
Double errors	4 %
Unsolved problems	3 %
Unclassifiable errors	2 %

Table 2. Percentage of correct solutions for the different types of simple numerical problems

Operations	Direct problems		Indirect problems	
	Type I	Type II	Type III	Type IV
Addition	98 %	79 %	94 %	94 %
Subtraction	96 %	75 %	93 %	68 %

Table 3. Collective pretest administered to experimental and control group in the teaching experiment*

Ten simple problems	
1) 2+4=.	6) .=3+7
2) 5+.=13	7) 15-9=.
3) .-9=8	8) .-1=11
4) 13-.=6	9) 15=.+8
5) .=12-8	10) .-7=5
Eight complex problems	
11) 7+3+.=17	15) .+5+6=13
12) .-6-1=9	16) .-4-5=3
13) 2+4+7=.	17) .+6+1=11
14) 20-.-12=4	18) 6-2=.+3

Two word problems

19) When Charles arrives at home in the evening, he has 6 marbles. Today, he lost 2 marbles. This morning Charles had ... marbles.

20) Peter gets 9 marbles from his mother. In the evening, he already has 17 of them. Today he won ... marbles.

* The same test was used as a posttest.

Table 4. Individual pretest administered to 6 pupils of the experimental and 3 pupils of the control group during the teaching experiment*

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1) "Correct or not" : 8+4=4+8

2) "Correct or not" : 8-4=4-8

3) "I will give you two problems. When you have answered the first problem, the result of the second one is very easy to find." (The first task is presented on a card; when the pupil has solved it, the outcome, which is written on a separate card, is also laid down. The second problem is then laid under the first.)

2+4+7= 13

.=2+4+7

4) "I will again give you two problems. The first one is very easy. The second one seems more difficult, but if you keep a sharp look out, you can find the solution quickly."

2+.=4 (not scored)

4+17=4+.

5) "Solve the following problem" 7=5+.+3 (unsolvable)

6) "Solve the following problem" 7-.=8 (unsolvable)

7) "Today you are the teacher, and you have to design tasks for your pupils. But, you are allowed to use only three numbers : 4, 7, and 3. Other numbers may not occur - neither in the statement of the task nor in the outcome."
An example is given using the numbers 2, 3, and 5. "Try to make as many problems as you can."
(The item is judged correct if three correct tasks are given, but it is scored wrong if more than one wrong task is proposed.)

8) 15=.+8

9) .-9=8

10) 20-.-12=4

---

* The same test was used as an individual posttest. All the items were presented to the children on cards.

Table 5. Results of the experimental and the control groups on collective pretests and posttests and on the retention test

Groups	Pretest	Posttest	Retention test
Experimental group	60 %	80 %	79 %
Control group	58 %	60 %	- *

* As mentioned before, the retention test was not administered to the control group.

Table 6. Number of technical and thinking errors of the experimental and the control group in the pretest and the posttest

Groups	Technical errors		Thinking errors	
	Pretest	Posttest	Pretest	Posttest
Experimental group	31	39	182	72
Control group	27	30	182	170

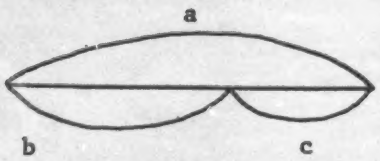
Table 7. Mean scores of the individually tested pupils of the experimental and the control groups on the individual pretest and posttest (maximum score : 10)

Groups	Pretest	Posttest
Experimental group (N=6)	3.3	7.2
Control group (N=3)	4.3	4.8

Figure 1. Schematic representation of the part-whole relation

.... - .... - ....

Figure 2. Example of the tasks used in the first lesson of TLU 3



a = ...
b = ...
c = ...

Figure 3. Example of the tasks used in the third lesson of TLU 3